Non-Diophantine arithmetics as a tool for formalizing information about nature and technology

Michele Caprio, Andrea Aveni and Sayan Mukherjee

Duke University, Durham, NC, USA; michele.caprio@duke.edu, andrea.aveni@duke.edu, sayan@stat.duke.edu

The theory of non-Diophantine arithmetics is based on a more general structure called an abstract prearithmetic. A generic abstract prearithmetic **A** is defined as

$$\mathbf{A} := (A, +_A, \times_A, \leq_A),$$

where $A \subset \mathbb{R}_+$ is the *carrier* of **A** (that is, the set of the elements of **A**), $\leq_A$ is a partial order on A, and $+_A$ and $\times_A$ are two binary operations defined on the elements of A. We conventionally call them addition and multiplication, but they can be any generic operation.

Abstract prearithmetic **A** is called *weakly projective* with respect to a second abstract prearithmetic $\mathbf{B} = (B, +_B, \times_B, \leq_B)$ if there exist two functions $g: A \to B$ and $h: B \to A$ such that, for all $a,b \in A$,

$$a +_A b = h(g(a) +_B g(b)) \qquad \text{and} \qquad a \times_A b = h(g(a) \times_B g(b)).$$

Function g is called the *projector* and function h is called the *coprojector* for the pair (**A**,**B**).

The *weak projection* of the sum $a +_B b$ of two elements of B onto A is defined as $h(a +_B b)$, while the weak projection of the product $a \times_B b$ of two elements of B onto A is defined as $h(a \times_B b)$.

Abstract prearithmetic **A** is called *projective* with respect to abstract prearithmetic **B** if it is weakly projective with respect to **B**, with projector f^{-1} and coprojector f. We call f, that has to be bijective, the *generator* of projector and coprojector.

Weakly projective prearithmetics depend on two functional parameters, g and h—one, f, if they are projective—and recover Diophantine arithmetic (the conventional arithmetic, called Diophantine from Diophantus, the Greek mathematician who first approached this branch of mathematics) when these functions are the identity. To this extent, we can consider non-Diophantine arithmetics as a generalization of the Diophantine one. A complete account on non-Diophantine arithmetics can be found in the recent book by Burgin and Czachor [4].

In this work, we consider three classes of abstract prearithmetics, $\{\mathbf{A}_M\}_{M\geq 1}$, $\{\mathbf{A}_{-M,M}\}_{M\geq 1}$, and $\{\mathbf{B}_M\}_{M\geq 0}$. These classes of prearithmetics are useful to describe some natural and computer science related phenomena for which the conventional Diophantine arithmetic fails. For example, the fact that adding one raindrop to another one gives one raindrop, or that putting a lion and a rabbit in a cage, one will not find two animals in the cage later on (cf. [2] and [5]).

They also allow avoiding the introduction of inconsistent Diophantine arithmetics, that is, arithmetics for which one or more Peano axioms were at the same time true and false. For example, in [1] Rosinger points out that electronic digital computers, when operating on the integers, act according to the usual Peano axioms for $\mathbb{N}$ plus an extra ad-hoc axiom, called the machine infinity axiom. The machine infinity axiom states that there exists $M \in \mathbb{N}$ far greater than 1 such that $M + 1 = M$. Clearly, Peano axioms and the machine infinity axiom together give rise to an inconsistency, which can be easily avoided by working with the prearithmetics we introduce.

In addition, $\{\mathbf{A}_M\}_{M\geq 1}$ and $\{\mathbf{A}_{-M,M}\}_{M\geq 1}$ allow to overcome the version of the paradox of the heap (or sorites paradox) stated in [3, Section 2]. The setting of this variant of the sorites paradox is adding one grain of sand to a heap of sand, and the question is, once a grain is added, whether the heap is still a heap.

We show that every element $\mathbf{A}_{M'}$ of $\{\mathbf{A}_M\}_{M\geq 1}$ is a complete totally ordered semiring, and it is weakly projective with respect to $\mathbf{R}_+$, the conventional Diophantine arithmetic of positive real numbers. Furthermore, we prove that the weak projection of any series $\Sigma_n a_n$ of elements of $\mathbb{R}_+ := [0,\infty)$ is convergent in each $\mathbf{A}_M$. This is an exciting result because it allows the scholar that needs a particular series to converge in their analysis to reach that result by performing a weak projection of the series onto $\mathbf{A}_M$, and then continue the analysis in $\mathbf{A}_M$.

The second class, $\{\mathbf{A}_{-M,M}\}_{M\geq 1}$, allows to overcome the paradox of the heap and is such that every element $\mathbf{A}_{-M',M'}$ is weakly projective with respect to the conventional real Diophantine arithmetic $\mathbf{R} = (\mathbb{R}, +, \times, \leq_{\mathbb{R}})$. The weak projection of any non-oscillating series $\Sigma_n a_n$ of terms in $\mathbb{R}$ is convergent in $\mathbf{A}_{-M',M'}$, for all $M' \geq 1$. The drawback of working with this class is that its elements are not semirings, because the addition operation is not associative.

The last one, $\{\mathbf{B}_M\}_{M\geq 0}$, is such that every element $\mathbf{B}_{M'}$ is a semiring and is projective with respect to the conventional real Diophantine arithmetic $\mathbf{R} = (\mathbb{R}, +, \times, \leq_{\mathbb{R}})$. The weak projection of any non-oscillating series $\Sigma_n a_n$ of terms in $\mathbb{R}$ is convergent in $\mathbf{B}_{M'}$, for all $M' \geq 0$. The drawback of working with this class is that its elements do not overcome the paradox of the heap.

References

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