Obtaining information about nature with finite mathematics

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Mathematical education at physics departments develops a belief that classical mathematics (involving infinitesimals, limits, continuity etc.) is the most fundamental mathematics, while finite mathematics is something inferior what is used only in special applications. And many mathematicians have a similar belief.

Historically it happened so because more than 300 years ago Newton and Leibniz proposed the calculus of infinitesimals, and since that time a titanic work has been done on foundation of classical mathematics. This problem has not been solved till the present time but for the majority of physicists and many mathematicians the most important thing is not whether a rigorous foundation exists but that in many cases standard mathematical technique works with a very high accuracy.

The idea of infinitesimals was in the spirit of existed experience that any macroscopic object can be divided into arbitrarily large number of arbitrarily small parts, and even in the 19th century people did not know about atoms and elementary particles. But now we know that when we reach the level of atoms and elementary particles then standard division loses its usual meaning and in nature there are no arbitrarily small parts and no continuity.

Therefore, a problem arises whether it is justified to use mathematics with infinitesimals although in nature there are no infinitesimals. Even from pure logical point of view, it seems strange to apply continuous mathematics for describing discrete (and even finite) nature. However, the logic of many physicists and mathematicians is as follows. The technique of continuous mathematics is very powerful and well elaborated, and we know that in many cases it works with a very high accuracy. So, something else should be used only when finiteness manifests in such a way that the accuracy given by continuous mathematics is not sufficient. I think that it is obvious that, sooner or later, we will encounter such cases. Moreover, as shown in my book (Lev, 2020), such cases already exist.

A typical situation in physics can be described by the following

Definition: *Let theory A contain a finite parameter and theory B be obtained from theory A in the formal limit when the parameter goes to zero or infinity. Suppose that with any desired accuracy theory A can reproduce any result of theory B by choosing a value of the parameter. On the contrary, when the limit is already taken then one cannot return back to theory A and theory B cannot reproduce all results of theory A. Then theory A is more general than theory B and theory B is a special degenerate case of theory A.*

Known examples are that classical nonrelativistic theory can be treated as a special degenerate case of relativistic one in the formal limit $c\to\infty$ (where c is the speed of light) and a special degenerate case of quantum mechanics in the formal limit $\hbar\to 0$ (where $\hbar$ is the Planck constant) while relativistic theory can be treated as a special degenerate case of de Sitter invariant theory in the formal limit $R\to\infty$ where R is the parameter of contraction from de Sitter groups or algebras to Poincare group or algebra.

In the literature those facts are explained from physical considerations but as explained in the famous Dyson's paper "Missed Opportunities" (Dyson, 1972), they follow from pure mathematical facts that symmetry groups of less general theories can be obtained from symmetry groups of more general ones by contraction. As argued in Ref. (Lev, 2020), quantum symmetries should be defined not by symmetry groups but by symmetry algebras and, as proved in this reference, the above statements can be proved on the language of Lie algebras as well.

As proved in Ref. (Lev, 2020), quantum theory based on classical mathematics is a special degenerate case of quantum theory based on finite mathematics in the formal limit $p\to\infty$, where p is the characteristic of the ring or field in the latter. So, by analogy with Dyson's approach, one can say that classical mathematics can be obtained from finite one by contraction $p\to\infty$. As a consequence:

Mathematics describing nature at the most fundamental level involves only a finite number of numbers, while the concepts of limit, infinitesimals and continuity are needed only in calculations describing nature approximately.

References:

Dyson F. (1972). *Missed Opportunities*. Bull. Amer. Math. Soc. **78**, 635-652 (1972).

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