Information in Numbers: Multiplicity and succession

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Numbers provide important information about the big diversity of various things. In particular, cardinal numbers give information about multiplicity and numerosity while ordinal numbers give information about succession and order.

In ancient Greece, the question regarding the ontic status of number was affirmed in the school of the Pythagoreans. In *number* they recognized the principle of existence of everything. Number does not only exist; it is the *essence* of what exists. This led to their well-known belief and premise: “Everything is number” (a mathesis universalis) Nevertheless, the relationship between a side and a diagonal of a regular pentagon would lead to the discovery of so-called “incomparable” (incommensurable) quantities – a discovery that the ordering (formative) role of “number” in the “cosmos” has been undermined, threatened by the *apeiron*, the unlimited-infinite. This discovery of irrational numbers was supposed to remain a secret, but Hippasos, who apparently made it public, was accused of "having betrayed a mathematical secret" (Riedweg 2005: 107).

What has become clear at this point is that although every numerical *ratio* is geometrically imaginable, not every relationship between two lines can be represented arithmetically. Laugwitz notes that it follows that Euclid treated number theory as part of geometry. He writes: “The relationship between numbers allows a geometric arrangement, but not every line relationship can be represented arithmetically. It bases the precedence of geometry over arithmetic, and the consequence of this can be found in the books of Euclid: The theory of numbers forms part of geometry.” The result was that Greek mathematics switched to another mode of explanation: number was replaced by the so-called geometricization of Greek mathematics.

Historically, there has indeed been a particular appreciation of what is usually referred to as the sequence of natural numbers, that is, the normal succession of numbers: 1, 2, 3, 4 and so on. The leading German mathematician at the end of the 19th century, Leopold Kronecker, an early intuitionist, formulated the following famous statement: “The Dear Lord Made the Whole Numbers; Everything Else is Human Work.” Stephen Hawking imitates Kronecker's words in the title of a book, “God Created the Integers, The Mathematical Breakthroughs that Changed History” (Hawking 2005) . In terms of current mathematical practice, Kronecker's statement actually refers to what are known as the natural numbers: (0), 1, 2, 3,… In mathematics, the number 0 has acquired an important role as addition identity, among others of the integers.

Although number with this lost its initial charm, the pendulum would later swing back again. The threat that lies locked up in the infinite has led to numerical considerations playing a subordinate role in medieval philosophical thought, because the school of Parmenides (with its emphasis on the static being) influences the hierarchy of beings with God as highest being (ipsum esse) resulting in (the “chain of being”).

After the arithmetization that Descartes brought about with his analytical geometry, it would not be long before the arithmeticism that originally led to the Pythagorean overestimation of number re-entered the battlefield. From Cauchy's textbook in analysis (published two hundred years ago in 1821) to the ultimate world of thought of Weierstrass, Cantor and Dedekind (at the end of the 19th century) we see how arithmeticism has mobilized anew its forces to reduce space to number. The difference between this 19th-century development and the original Pythagorean arithmeticism lies in the fact that where it was infinity that gave rise to the geometricization of Greek mathematics, in the 19th century it was again the infinite that this time accompanied the reverse development moving towards a seemingly complete arithmetization of mathematics.

Greek mathematics, however, has only accepted what was known as the potentially infinite, while Weierstrass, Dedekind, and Cantor have utilized the actual infinite. These terms are derived from Aristotle. Later, the two forms of the infinite would also be referred to as the incomplete infinite and the completed infinite. Cantor himself described his view of the nature of these two kinds of infinity in the “Communications on the Doctrine of the Transfinite” as follows: “The potentially infinite is preferably indicated where an indefinite, variable finite quantity occurs, which either increases over all boundaries [...], or decreases below all boundaries in smallness ”; "By the actual infinity, on the other hand, is meant a quantum, which on the one hand is not variable, but rather fixed and determined in all parts, is a real constant, and at the same time on the other hand surpasses any similar finite magnitude in size" (Cantor 1962: 401 – originally in Cantor 1887–1888). In the description he gives in a letter from 1886 the actual infinity is additionally qualified with the term *festes* (*fixed*): The actual infinity always refers to a constant quantum, which is fixed in itself (“ein in sichfestes Quantum ”) (from the Letter Book included in Meschkowski 1967: 249).

With the development of the set theory (on the basis of the utilization of actual infinity) the problem question regarding the status of number was only further complicated, because then we not only had to talk about number, but also about sets of numbers, that is say numerical sets. This raises a new question, namely: What is a set? The well-thought-out description that Cantor gives of a collection in 1895 reads: “By a 'set' I mean every summary M of certain, well-distinguished objects *m* of our intuition or our thinking (which are called the 'elements' of M becomes) into a whole” (Cantor 1895–1897:481; Cantor 1962:282). What immediately stands out in this definition is that it does two things at once, namely to point at a discrete quantity, with its more and less, and to introduce the idea that it is given *at once* (simultaneously as one multiplicity).

Cantor's definition thus makes use of both our sense of number and our sense of space at the same time. After set theory has been axiomatized, however, it has become apparent that even a set theoretical foundation of natural numbers had to assume the infinity axiom (cf. Meschkowski (1967:192)) which further inspired the intuitionist mathematicians to prioritize the arithmetical order of succession (underlying the principle of mathematical induction) as the non-formalizable starting point for mathematics. Additionally, each axiomatization assumes primitive (undefined) terms, while Gödel even points out that every attempt terms nature of a set gets stuck in a circular reasoning if it wants to be defined.

In the subsequent part of the presentation on multiplicity and succession will be related to the difference between the successive infinite and the at once infinite. Russell emphasizes that progressions are the essence of discreteness and that the whole-parts relation is primitive. A systematic account of infinite totalities will be given.